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RESEARCH ARTICLE

A computational study on general equilibrium pricing of derivative securities

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Abstract This paper analyses the accuracy of replicating portfolio methods in predicting asset prices. In a two-period, general equilibrium model with incomplete financial markets and heterogeneous agents, a computational study is conducted under various distributional assumptions. The focus is on the price of a call option on an underlying risky asset. There is evidence that the value of the (approximate) replicating portfolio is a good approximation for the general equilibrium price for CRRA preferences, but not for CARA preferences. Furthermore, there is strong evidence that the introduction of the call option reduces market incompleteness, but that the price of the underlying asset is unchanged. There is, however, inconclusive evidence on the welfare effects of the option.

Keywords Asset pricing · General equilibrium · Incomplete markets

JEL Classification Cation: D52 · G12

1 Introduction

Ever since the seminal contributions of Black and Scholes (1973), Merton (1973) and Harrison and Kreps (1979), there has evolved an entire industry concerned with asset

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pricing based on martingale methods. These methods are based on the two fundamental theorems of finance, the first of which states that, in the absence of arbitrage opportunities, there exists a probability measure under which each asset's price is the expectation of the future dividends. The second fundamental theorem then states that this measure is unique in complete markets. In a two-period economy with finitely many future states and one consumption good, the equivalent martingale measure can be interpreted as a vector of state prices, representing the price of consumption in each uncertain state in future.

In essence, financial markets are complete if the economy is isomorphic to an economy with a complete set of contingent contracts (a so-called Arrow–Debreu economy, see Debreu 1959, Chap. 7). Completeness allows the market to determine asset prices in such a way that all agents' present value vectors are equalised. The present value vector of an agent lists her individual valuation of each future state (in terms of the consumption good) given all asset prices.

From a practical point of view the two fundamental theorems are extremely useful. If markets are complete then, in order to value a (new) financial asset, no knowledge whatsoever about preferences or initial endowments is needed. In complete markets, namely, any new asset is redundant and its no-arbitrage value can be computed as a linear combination of the prices of the existing assets. The weights are determined by the orthogonal projection of the dividend stream of the new asset on the market span and are called the replicating portfolio. The resulting price is called the *replicating portfolio value* (RPV).

In incomplete markets, however, the present value vectors are, generically, not equal in equilibrium. Their orthogonal projections on the market span, however, are. In other words, there are several valid vectors of state prices in equilibrium. This happens because the dimension of the subspace orthogonal to the market space is larger than one. This, in turn, implies that new assets can, generically, not be expected to be redundant.[1] If a new asset is non-redundant no replicating portfolio exists. The best one can hope for is to find an approximate replicating portfolio via an orthogonal projection on the market space and, therefore, an approximate value for the new asset. This method has been advocated by, for example, Föllmer and Sondermann (1986).[2]

There are two reasons why the RPV might lead to structural errors in predicting a new asset's value in incomplete markets. Firstly, market incompleteness leads to non-existence of an exact replicating portfolio, as mentioned before. So, the portfolio used to compute the RPV is subject to approximation error. Secondly, the introduction of a new asset might change the prices of all other traded assets in equilibrium as well, due to the resulting change in the market span. This then leads to an error due to the fact that the "wrong" asset prices are used to determine the RPV.[3] Indeed, there is

[1] In fact, if markets are sufficiently incomplete, no redundant options exist at all (see Baptista (2007) and references therein).

[2] Whether or not actual financial markets are incomplete is open for debate. However, even if markets are, in fact, complete, asset valuation typically only uses a subset of all traded assets. That is, asset valuation then takes place in essentially incomplete markets (see Ross (2005) for an elaboration of this point).

[3] This is reminiscent of the "Lucas-critique" in macroeconomics, see Lucas (1976).

empirical evidence that the introduction of new batches of options has substantially changed asset prices in the past (see Conrad 1989; Detemple and Jorion 1990). Several theoretical papers have been devoted to this topic. Weil (1992) and Elul (1997), for example, show that the introduction of a new asset permits agents to better share risk. This weakens the need for precautionary savings and, hence, leads to a higher interest rate. This, in turn, reduces the prices of all assets in the economy. Oh (1996), on the other hand, shows that in economies with mean-variance, or CARA preferences and normally distributed asset payoffs, the price of any risky asset relative to the riskless bond is unaffected by changes in the market span. Calvet et al. (2004), however, show that in a CARA-normal economy with limited participation relative prices are in fact influenced by financial innovation.

This paper conducts a computational study to assess these issues. A two-period economy with three heterogeneous agents is studied. On the financial market two assets, a riskless bond and a risky asset are traded. Under several distributional assumptions we compare the performance of RPV as a predictor for the general equilibrium price of a call option on the risky asset for different strike prices. Agents' utility functions are either of the CARA or CRRA type. We also study whether the prices of the existing assets change substantially after the introduction of the call option.

First of all, it is found that the introduction of the call option significantly increases the degree of market completeness, which is measured by the fraction of the variance of total initial endowments that can be traded on the market. The findings also indicate that, if agents have CRRA preferences, the RPV is a relatively good predictor for the general equilibrium value (GEV) of the option. If, however, agents have CARA preferences, then the RPV is significantly different form the GEV. It is argued that this is due to the fact that CRRA agents care solely about relative changes in consumption, whereas CARA agents care only about absolute changes. Interestingly, this is exactly the property which leads many researchers studying (calibrated) infinite horizon incomplete market models to use a CARA specification for tractability (cf. Angeletos and Calvet 2006 and references therein). This study shows that (i) CRRA preferences might provide a better defense for the use of a replicating portfolio argument and (ii) even in the simplest two-period setting with incomplete markets, CRRA and CARA preferences lead to different conclusions.

Furthermore, the price of the underlying asset does not significantly change after the introduction of the option. Given the stylised facts on the reaction of prices of underlying assets to the introduction of options, this indicates that one has to be careful in using two-period models for option pricing. Finally, we find inconclusive evidence on the effect of the introduction of the option on agents' welfare, as measured by the equivalent variation. In most cases we find that it is not statistically significantly different from zero.

The paper is organised as follows. In Sect. 2, the two-period general equilibrium model with incomplete financial markets (GEI) is introduced. In Sect. 3 the replicating portfolio approach is described in detail and in Sect. 4 the computational study is presented. Section 5 discusses the simulation results, while Sect. 6 concludes.

2 The GEI finance economy

In the economy there are two time periods, $t = 0, 1$, where $t = 0$ denotes the present and $t = 1$ denotes the future. At $t = 0$ the state of nature is known to be $s = 0$. Uncertainty over possible states of nature at $t = 1$ is modelled by a probability space $(S, \mathcal{S}, P)$, where S is assumed to be a finite set indexed (with slight abuse of notation) by $s = 1, \ldots, S$. There are $H \in \mathbb{N}$ agents (or investors or households) indexed by $h = 1, \ldots, H$. There is one consumption good, which can be interpreted as income. A consumption plan for agent $h \in \{1, \ldots, H\}$ is a vector $x^h \in \mathbb{R}_+^{S+1}$, where x_s^h gives the consumption level in state $s \in \{0, 1, \ldots, S\}$.[4]

Each agent $h = 1, \ldots, H$, is characterised by a vector of initial endowments, $\omega^h \in \mathbb{R}_+^{S+1}$, and a utility function $u^h : \mathbb{R}_+^{S+1} \to \mathbb{R}$. We denote aggregate initial endowments by $\omega = \sum_{h=1}^{H} \omega^h$. It is assumed that the vector of aggregate initial endowments is strictly positive, i.e. $\omega \in \mathbb{R}_{++}^{S+1}$. Furthermore, for each agent $h = 1, \ldots, H$, it is assumed that the utility function, u^h, is continuous, strictly monotone and strictly quasi-concave on $\mathbb{R}_+^{S+1}$. These assumptions ensure that (i) in each period and in each state of nature there is at least one agent who has a positive amount of the consumption good and (ii) each agent's demand for assets is a continuous function.

The market for the consumption good is a spot market. The agents can smoothen consumption over time and uncertain states by trading on the financial market, where $J \in \mathbb{N}$ financial contracts (or assets) are traded, indexed by $j = 1, \ldots, J$. The future payoffs (dividends) of these assets are collected in a matrix

$$A = [d_s^j]_{s=1,\ldots,S}^{j=1,\ldots,J} \in \mathbb{R}^{S \times J},$$

where d_s^j is the payoff of one unit of asset j in state s. The matrix A is assumed to be of full rank, implying that there are no redundant assets, i.e. assets whose dividends can be replicated by a linear combination (portfolio) of the other assets. The market subspace is denoted by $\langle A \rangle = Span(A)$. That is, the market subspace consists of those income streams that can be generated by trading on the financial markets. If $S = J$, the market subspace consists of all possible income streams and markets are said to be complete. If $J < S$, there is idiosyncratic risk and markets are incomplete.

A *GEI economy* is defined as a tuple $\mathcal{E} = \Big((u^h, \omega^h)_{h=1,\ldots,H}, A\Big)$. Given a GEI economy $\mathcal{E}$, agent h can trade assets by buying a portfolio $\theta^h \in \mathbb{R}^J$ given the (row)vector of prices $q \in \mathbb{R}^J$.[5] The budget set for agent $h = 1, \ldots, H$ then equals

$$B^h(q) = \Big\{ x \in \mathbb{R}_+^{S+1} \Big| \exists_{\theta \in \mathbb{R}^J} : x_0 - \omega_0^h \leq -q\theta, \mathbf{x_1} - \boldsymbol{\omega}_{\mathbf{1}}^h = A\theta \Big\}. \tag{1}$$

Given the asset payoff matrix A we will restrict attention to asset prices that generate no arbitrage opportunities, i.e. asset prices q such that there is no portfolio generating a

[4] In general a vector $x \in \mathbb{R}^{S+1}$ is denoted $x = (x_0, \mathbf{x_1}) \in \mathbb{R} \times \mathbb{R}^S$ to separate x_0 in period $t = 0$ and $\mathbf{x_1} = (x_1, \ldots, x_S)$ in period $t = 1$.

[5] We follow the convention of denoting prices in row vectors and quantities in column vectors.

non-negative income stream. It can be shown that the set of no-arbitrage prices equals (cf. Magill and Quinzii 1996)

$$Q = \{q \in \mathbb{R}^J | \exists_{\pi \in \mathbb{R}^S_{++}} : q = \pi A\}. \tag{2}$$

The vector $\pi \in \mathbb{R}^S_{++}$ can be interpreted as a vector of state prices. So, the no-arbitrage price for security j equals the present value of security j given some vector of positive state prices π.

In complete markets state prices are uniquely determined up to normalisation.[6] If one normalises state prices on the unit simplex, π can be interpreted as a probability measure. Since no-arbitrage prices are simply the expected value of asset payoffs under π, this probability measure is usually referred to as the *martingale measure*. In complete markets only the standard assumptions on utility functions are needed to price any asset without having to resort to any equilibrium argument. If markets are incomplete, however, π is not uniquely determined. This is exactly the reason why asset pricing in incomplete markets is conceptually much more difficult.

Under the assumptions made on utility functions, the demand function $x^h(q) = \arg\max\{u^h(x)|x \in B^h(q)\}$, is well-defined for all $h = 1, \ldots, H$, and all $q \in Q$. It can easily be shown that $x^h(q)$ and the security demand function, $\theta^h(q)$, determined by $A\theta^h(q) = x^h_{\mathbf{1}}(q) - \omega^h_{\mathbf{1}}$, are continuous on Q. A *financial market equilibrium* (FME) for a GEI economy $\mathcal{E}$ is a tuple $\left((\bar{x}^h, \bar{\theta}^h)_{h=1,\ldots,H}, \bar{q}\right)$ with $\bar{q} \in Q$ such that:

1. $\bar{x}^h \in \arg\max\{u^h(x^h)|x^h \in B^h(\bar{q})\}$ for all $h = 1, \ldots, H$;
2. $A\bar{\theta}^h = \bar{x}^h_{\mathbf{1}} - \omega^h_{\mathbf{1}}$ for all $h = 1, \ldots, H$;
3. $\sum_{h=1}^H \bar{\theta}^h = \Theta$,

where $\Theta \in \mathbb{R}^J_+$ denotes each asset's net-supply. For any GEI economy $\mathcal{E}$, satisfying the assumptions introduced above, there exists a FME on Q. [cf. Hens (1991) or Talman and Thijssen (2006)].

3 The value of a new financial asset

Let $(S, \mathcal{S}, P)$ be a (discrete) probability space and let $\mathcal{E} = (u, \omega, A)$ be a two-period GEI economy, with J assets. Suppose that a new asset is introduced with future dividends $d^{J+1} \in \mathbb{R}^S$. This new asset can be a new financial product which is actually going to be traded on the financial market, like a derivative security. It could also represent the risky payoffs of a real investment project of a firm. In this case the asset will not actually be traded, but the firm might wish to value the project as the shareholders would if the project were traded.[7]

In complete markets the new asset is redundant and its payoff can, hence, be written as a linear combination, θ, of the columns in A. The vector θ is called the *replicating portfolio* and the value of the redundant asset should equal θq, where q is the vector of

[6] The space orthogonal to $\langle A \rangle$ is one-dimensional.

[7] See Dixit and Pindyck (1994) for an introduction to the valuation of real investment projects using financial option pricing techniques.

prices of the J assets in A. Generically, however, if financial markets are incomplete, then no unique replicating portfolio exists. Following Föllmer and Sondermann (1986) one could use the orthogonal projection of d^{J+1} on $\langle A\rangle$ instead. Let $\theta_A(d^{J+1})$ denote the (unique) replicating portfolio of $d^{J+1}_{\langle A\rangle}$, where $x_{\langle A\rangle}$ denotes the projection in $||\cdot||_2$ of $x \in \mathbb{R}^S$ onto $\langle A\rangle$. The value of this *approximate replicating portfolio*, denoted by $RPV(d^{J+1})$, is then

$$RPV(d^{J+1}) = q\theta_A(d^{J+1}). \tag{3}$$

From a practical point of view the RPV approach is appealing, since $\theta_A(d^{J+1})$ is essentially obtained by performing a linear regression of d^{J+1} on the existing assets in A. The matrix A would then consist of past observations on the dividends of the J stocks. If the new asset is, for example, a derivative security, which is written on one of the assets in A, then d^{J+1} can be computed for all observations, and $\theta_A(d^{J+1})$ can be obtained. If the derivative security is redundant, then the aforementioned regression should have $R^2 = 1$. So, $R^2 < 1$ is an indication of non-redundancy of the new asset.

The replicating portfolio approach computes the expectation of the price of the new asset, conditional on the existing market span, using risk-neutral probabilities that are derived from market prices. A clear advantage of this procedure is that one only uses observable dividends and prices and no unobservables like preferences and endowments. However, if $d^{J+1} \notin \langle A\rangle$, then agents in the economy will, typically, not agree on the valuation of the residual of d^{J+1} (the orthogonal projection of d^{J+1} on the orthogonal complement of $\langle A\rangle$).[8] Also, RPV assumes that in the economy with the new asset the prices of the existing assets do not change. There is, however, no guarantee that this will indeed be the case.

In order to account for market incompleteness and changes in the market span, $\langle A\rangle$, let $\tilde{A} = \left[A \ d^{J+1}\right] \in \mathbb{R}^{S\times(J+1)}$ be the asset payoff matrix *after* the new asset has been introduced in the market. Note that, if initial endowments include asset holdings and the new security is introduced in non-zero net-supply, then initial endowments change as well to, say, $\tilde{\omega}$. That is, after a security with payoffs d^{J+1} has been introduced, the new GEI economy is $\tilde{\mathcal{E}} = (u, \tilde{\omega}, \tilde{A})$. Let $\tilde{q} \in \mathbb{R}^{J+1}$ be a vector of equilibrium prices in this economy. Then the general equilibrium value of the new asset, denoted by $GEV(d^{J+1})$, is

$$GEV(d^{J+1}) = \tilde{q}_{J+1}. \tag{4}$$

The main difference with the approximate replicating portfolio value is that the RPV values the new asset in the economy $\mathcal{E}$, whereas the GEV values the asset in the economy $\tilde{\mathcal{E}}$.

To illustrate how the RPV can provide a poor proxy for the GEV, consider an economy with two agents, three states of nature and one asset. The asset is a contingent contract on the second state, i.e. $A = [e_2]$, where e_i is the i-th unit vector. So, $e_2 \in \mathbb{R}^3$ is a basis for $\langle A\rangle$. Initial endowments are taken to be $\omega^1 = (2, 1, 1, 1)$

[8] See, for example Magill and Quinzii (1996) for an extensive discussion.

and $\omega^2 = (1, 2, 2, 2)$. Suppose that both agents have identical, time-separable von Neumann–Morgenstern utility functions with constant relative risk aversion (CRRA),

$$u^h(x_0, x_{\mathbf{1}}) = \log(x_0) + 0.95 \sum_{s=1}^{3} \frac{1}{3} \log(x_s).$$

After normalising the price of date 0 consumption to 1, the equilibrium price for the asset in this economy is $q = 0.3246$. Suppose now that a contingent contract for the third state is introduced, i.e. $\tilde{A} = \left[e_2 \; e_3\right]$. In this case $\{e_2, e_3\}$ is a basis for $\langle \tilde{A} \rangle$. Since the dividend stream of the new asset is orthogonal to $\langle A \rangle$, it immediately follows that $\theta_{\langle A \rangle}(e_3) = 0$, implying that $RPV(e_3) = 0$. Given the same endowments, the equilibrium prices of the two assets are $\tilde{q} = (0.3221, 0.3221)$. This example makes clear that RPV can be a bad approximation of the true value of the asset. It also shows that the prices of existing assets can change when a new asset is introduced.

This example is, of course, an extreme case. Most financial innovations will not be orthogonal to the existing market span. It does show, however, that in general it will be difficult to say how accurate the RPV predicts the value that the market will assign to a financial innovation. In the next section we will, therefore, conduct some simulation experiments to asses the accuracy of RPV for the introduction of a particular financial innovation, namely a call option on an existing risky asset.

4 A simulation analysis

In this section, the two valuation methods presented in Sect. 3 are numerically assessed, under several distributional assumptions, for the introduction of a call option on the J-th asset. We are interested in a number of empirical questions, the first of which is the accuracy of $RPV(d^{J+1})$ in predicting $GEV(d^{J+1})$. We also want to assess whether the equilibrium price of the underlying asset changes significantly. This would have important consequences in itself for the practise of derivative pricing. If, namely, the price of the underlying asset changes significantly with the introduction of an option, then the validity of the standard assumption of an exogenous and unchanging underlying asset price process (like a geometric Brownian motion) might be questioned.

Apart from the market value of a financial innovation, however, an important question is how much better off the agents in the economy are after the introduction of the new asset. In standard microeconomic models one often uses the equivalent or compensating variation to measure effects on agents' wealth of price and/or income changes. Since the equivalent variation takes current prices—asset prices before the introduction of the new asset—as the starting point this seems the better approach in the current setting. Given an equilibrium $\left((\bar{x}, \bar{\theta}), \bar{q}\right)$ in $\mathcal{E}$ and an equilibrium $\left((\tilde{x}, \tilde{\theta}), \tilde{q}\right)$ in $\tilde{\mathcal{E}}$, the equivalent variation of the new asset for agent h, denoted by $EV^h(d^{J+1})$, is defined as

$$EV^h(d^{J+1}) = e^h(\bar{q}, \tilde{u}^h) - \bar{q}\bar{\theta}^h,$$

where for all h, $e^h(\bar{q}, \tilde{u}^h)$ solves

$$e^h(\bar{q}, \tilde{u}^h) = \min_{\{\theta \in \mathbb{R}^J | \omega^h + W\theta \geq 0\}} \{\bar{q}\theta | u^h(\omega^h + W\theta) \geq \tilde{u}^h\},$$

where $W = \begin{bmatrix} -\bar{q} \\ A \end{bmatrix}$, and $\tilde{u}^h = u^h(\tilde{x}^h)$. That is, $EV^h(d^{J+1})$ measures the total amount agent h should receive (in addition to her initial endowments) in $\mathcal{E}$ to be at least as well off as in $\tilde{\mathcal{E}}$.

Markets are incomplete when $J < S$. In itself, however, this does not give an indication of the degree of market incompleteness. The possibilities of agents to trade their initial endowments, for example, give a better indication. Indeed, if an agent's endowments lie in the market space, $\omega^h \in \langle A \rangle$, then for this agent, markets are *effectively* complete, because her endowments are uniquely priced. Following Calvet et al. (2004), we define the degree of market completeness of the economy $\mathcal{E}$ as the fraction of variability in total endowments that is traded on the market, i.e.

$$\alpha(\mathcal{E}) = \frac{Var(\omega_{\langle A \rangle})}{Var(\omega)}.$$

This leads to the following quantities of interest. Let q and $\tilde{q}$ be equilibrium price vectors in the economies $\mathcal{E}$ and $\tilde{\mathcal{E}}$, respectively. For all simulations, we will test the following hypotheses,

$$H_0: \quad \alpha(\mathcal{E}) = \alpha(\tilde{\mathcal{E}}), \tag{5}$$

$$H_0: \quad EV^h(d^{J+1}) = 0, \text{ all } h = 1, \ldots, H, \tag{6}$$

$$H_0: \quad q_j = \tilde{q}_j, \text{ all } j = 1, \ldots, J, \tag{7}$$

$$H_0: \quad GEV(d^{J+1}) = RPV(d^{J+1}), \tag{8}$$

The first hypothesis tests whether market incompleteness changes significantly by the introduction of the call option on the risky asset. The second set of hypotheses test whether the agents in the economy are actually significantly better of with the new asset. The third hypothesis tests whether the equilibrium price of the underlying asset changes significantly. Finally, the last hypothesis tests whether the general equilibrium value and the (approximate) replicating portfolio value are significantly different. For each of these tests we compute the asymptotically normal z-statistic.

4.1 General simulation set-up

It is assumed that there are $J = 2$ assets in the economy. The first asset is a riskless bond with payoff stream $d^1 = 1_S$, which is in zero supply. The second asset is a risky asset with

$$d^2 = \beta f + (1 - \beta)\varepsilon,$$

where $f \in \mathbb{R}^S$ is an economy-wide factor, $\varepsilon \in \mathbb{R}^S$ is an asset-specific shock, and $\beta \in [0, 1]$ is the asset's exposure to the economy-wide factor. Throughout, β is drawn randomly from the interval [0.25,0.75]. The risky asset is assumed to be in unit supply.

The economy consists of $H = 3$ agents with time-separable von Neumann–Morgenstern utility functions

$$u^h(x_0, x_{\mathbf{1}}) = v^h(x_0) + \delta^h \sum_{s=1}^{S} p_s^h v^h(x_s),$$

where δ^h is the discount rate of agent h and p_s^h is the probability which agent h assigns to state s. Throughout, we take $\delta^h = 0.95$ and, unless otherwise stated, $p_s^h = 1/S$, all $h = 1, \ldots, H$ and $s = 1, \ldots, S$. The function $v(\cdot)$ is either of the CARA or the CRRA type, i.e.

$$v^h(x) = 1 - e^{-\gamma^h x}, \quad \text{or} \quad v^h(x) = \frac{x^{1-\gamma^h} - 1}{1 - \gamma^h},$$

respectively, where the rate of risk aversion, γ^h, is randomly drawn from $\{1, \ldots, 6\}$, all $h = 1, \ldots, H$.

Initial endowments consist of labour income (wages) and dividend income. For agent h, labour income is equal to

$$L^h = \zeta^h f + (1 - \zeta^h) l^h,$$

where $l^h \in \mathbb{R}^S$ is an agent specific wage shock and ζ^h is the exposure of agent h's wages to the economy-wide factor. Throughout we take $\zeta^h = 0.5$, all $h = 1, \ldots, H$. The initial portfolios, θ_0^h, of risky asset holdings are $\theta_0^1 = 0$, $\theta_0^2 = 1/3$, and $\theta_3^2 = 2/3$. Endowments are assumed to consist for 2/3 of wages and for 1/3 of asset income, i.e.

$$\omega_{\mathbf{1}}^h = \frac{2}{3} L^h + \frac{1}{3} d^2 \theta_0^h.$$

Date 0 initial endowments are given by $\omega_0 = \begin{bmatrix} 2/3 & 1 & 4/3 \end{bmatrix}$.

We consider the valuation of a call option on the risky asset, i.e.

$$d^3 = \max\{0, d^2 - d_K^2 1_S\},$$

where the strike price, d_K^2, is the K-th percentile of d^2. Finally, every run consists of 100 simulations. All equilibria are computed using a differentially implementable homotopy algorithm, constructed by Herings and Kubler (2002).[9]

[9] See Appendix A for a description of the algorithm.

4.2 Tri-nomial shocks

As a first model, we study economies where the factor and the asset and wage specific shocks form a tri-nomial tree. That is, we take

$$
\begin{aligned}
f &\in \{0.89, 1.02, 1.15\} \\
\varepsilon &\in \{0.89, 1.02, 1.15\} \\
l^h &\in \{0.92, 1.02, 1.12\} \quad \text{all } h = 1, \ldots, H.
\end{aligned}
$$

This implies that $\mathbb{E}(d^2) = \mathbb{E}(L^h) = 1.02$, which represents an average 2% increase in wages and a dividend rate of 2%. Also, $\sigma(L^h) = 0.125$, where $\sigma(\cdot)$ denotes the standard deviation. The variance of d^2 depends on the, randomly generated, factor loading β. For all simulations we take $K = 0.5$.

The results of the tests (5)–(8) are reported in Table 1 of Appendix B. In both cases the introduction of the option does not change the degree of market completeness (5) significantly. However, $GEV(d^3)$ is significantly different from $RPV(d^3)$ in both cases (at the 1 and 10% level for CARA and CRRA, respectively). The equivalent variation is only significant (at the 1% level) for agent 3 (the richest agent) in the CRRA simulations. This could indicate that the additional utility that agents derive from better opportunities for risk-sharing due to the presence of the option are largely offset by the price they pay for this asset. Only agent 3 benefits significantly, possibly because this agents finds herself to be the provider of consumption smoothing possibilities over risky states to the other, less well-off, agents. Finally, in either case, the price of the underlying asset does not significantly change after the introduction of the option. Only for the CRRA case does the price of the riskless asset change significantly (at the 10% level).

4.3 Uniform shocks

In the second model, we study economies where the factor and the asset and wage specific shocks are uniformly distributed. That is, we take

$$
\begin{aligned}
f &\sim U(0.8, 1.24) \\
\varepsilon &\sim U(0.8, 1.24) \\
l^h &\sim U(0.85, 1.2) \quad \text{all } h = 1, \ldots, H.
\end{aligned}
$$

Again, this implies that $\mathbb{E}(d^2) = \mathbb{E}(L^h) = 1.02$ and that the variance of labour income is lower than the variance of asset income. In the finite setting that is considered in this paper, these distributions are achieved by simulating 500 observations on f, ε, and l^h, according to the distributions specified above.[10] For these, and the following, simulations the option's strike price K is drawn randomly from the interval [0.25,0.75].

[10] That is, $S = 500$.

The results of the tests (5)–(8) are reported in Table 2 of Appendix B. In both cases, the introduction of the option changes the degree of market completeness (5) significantly at the 1% level. However, $GEV(d^3)$ is significantly different from $RPV(d^3)$ (at the 1% level) only for the CARA specification. The equivalent variation is now only significant (at the 1% level) for agents 1 and 2 in the CARA specification, but not for agent 3 at all. Finally, for either utility function, the price of the underlying asset does not significantly change after the introduction of the option. Only for the CRRA case does the price of the riskless asset change significantly (at the 5% level).

4.4 Log-normal shocks

In the third model, we study economies where the factor and the asset and wage specific shocks are log-normally distributed. We simulate, as before, 500 draws for f, ε, and l^h, where

$$f \sim LN(1.02, 0.01) \tag{9}$$

$$\varepsilon \sim LN(1.02, 0.01) \tag{10}$$

$$l^h \sim LN(1.02, 0.0025) \text{ all } h = 1, \ldots, H. \tag{11}$$

Again, we assume that all random variables grow at an average rate of 2%, with a smaller variance for labour income than dividends.

All empirical results are reported in Table 3 of Appendix B. In both cases, the introduction of the option changes the degree of market completeness (5) significantly at the 1% level. However, $GEV(d^3)$ is significantly different from $RPV(d^3)$ (at the 1% level) only for the CARA specification. In this model, the equivalent variation is now significant at the 1% level for agent 1 in both specifications, and at the 5% level for agent 3 only for the CRRA specification. This time, the equivalent variation for agent 1 is insignificant for both specifications. Finally, for either utility function, the prices of both the riskless asset and the underlying asset do not change significantly after the introduction of the option.

4.5 Log-normal shocks with downward jumps

One implication of assuming preferences exhibiting risk aversion is that agents not only care about mean and variance, but also about higher moments. In order to ascertain the influence of such higher moments, we study simulations where the dividends and wages are skewed. This is achieved in the following way. After drawing 500 observations for f, ε, and l^h according to (9)–(11), respectively, we randomly draw 25 observations of f, which we replace by the smallest draw of f. This then represents a log-normal distribution with downward jumps.

All empirical results are reported in Table 4 of Appendix B. The degree of market completeness again changes significantly at the 1% level. Also, most equivalent variations are, again, insignificant. The general equilibrium value, $GEV(d^3)$, is significantly different from $RPV(d^3)$ at the 1% level only for the CARA specification, and the change in the price of the underlying asset is insignificant. At the 10% level, the price change for the riskless asset in the CARA specification is significant.

4.6 Heterogeneous beliefs

Finally, we look at the influence of heterogeneous beliefs. So far, it has been assumed that all agents believe that all states $s = 1, \dots, S$, are equally likely. In this simulation run we allow beliefs to be heterogeneous. Wages and dividends are constructed in the same way as in the previous model. In addition, three different types of beliefs are assumed: uniform, pessimistic, and optimistic. An agent with uniform beliefs attaches equal probabilities to all states of nature. An agent with pessimistic beliefs assigns higher probabilities to states of nature with a low outcome. This is achieved in the following way. The 33% of states with the lowest 33% of values for the factor f are assigned a probability twice as high as the 33% of states with the highest 33% of values for f. An agent with optimistic beliefs constructs beliefs just the other way around. In every simulation round and for every agent, the belief type is drawn randomly.

All empirical results are reported in Table 5 of Appendix B. The degree of market completeness again changes significantly at the 1% level. For the CRRA specification, all agents equivalent variations are significantly different from zero (at the 5 or 1% level). The general equilibrium value, $GEV(d^3)$, is significantly different from $RPV(d^3)$ at the 1% level only for the CARA specification. The changes in the prices of both the riskless asset and the underlying asset are insignificant.

5 Discussion

Based on the simulations, several observations can be made on the effects in the model economy of the introduction of new financial assets. We can also say something on the robustness of the (approximate) replicating portfolio method as a proxy for the general equilibrium value of a new asset in economies with heterogeneous agents and incomplete financial markets. A first observation is that, apart from the tri-nomial model, the introduction of a call option on the risky asset significantly changes (at the 1% level) the degree of market completeness as defined in (5). Since all z_α are positive, this implies that in an economy with risk-averse agents, the new asset should have positive value. If, however, the welfare change of the agents is assessed using the equivalent variation, no clear picture for this value emerges. Only in some models, for some preference specifications and for some agents the equivalent variation is significantly different from zero. This implies that the price for additional consumption smoothing over risky states that the new asset provides is such that no agent actually makes a profit on it; not even the richest agent. This might happen because all agents are risk averse and, therefore, have a positive demand for additional consumption smoothing. Since the option is in zero supply, however, the richer agent will provide this product to the poorer ones, but needs a higher price to compensate for the loss of her own consumption smoothing resulting from selling the option (i.e. from taking on more risk).[11]

Secondly, the price of the existing risky asset does not significantly change due to the introduction of the new asset, even though the new asset is not perpendicular to the existing risky asset. In other words, the demand for the risk sharing opportunities provided by the risky asset remains unchanged. Changes in the price of the riskless asset are also statistically insignificant in almost all models. From this it can be concluded that in economies with incomplete markets and heterogeneous risk averse agents, the demand for a new asset is solely determined by the *additional* market span it provides. It also indicates that the two-period GEI model cannot explain the stylised fact that the introduction of options does have a significant impact on the price of the underlying (cf. Conrad 1989; Detemple and Jorion 1990). This might be due to the fact that the option's payoff in a two-period model is—by necessity—exogenous, whereas an important function of options is to insure against endogenous changes in the equilibrium price of the underlying. This would suggest that options can only be meaningfully analysed in models with at least three periods, where the option's exercise time is not the terminal date of the model. To my knowledge the only attempt in the literature to model option payoffs endogenously has been made by Polemarchakis and Ku (1990).

Thirdly, for all distributions, the CARA specification leads to significantly different general equilibrium values compared to the RPV (at the 1% level). For the CRRA specification, however, the difference is not significant (at the 1% level) for any distribution. This seems to indicate that RPV is a better predictor for the equilibrium value of the option for CRRA than for CARA preferences. Because of the general equilibrium nature of the model it is difficult to understand this phenomenon in full generality. A partial equilibrium approach, however, can shed some light on the underlying differences between CARA and CRRA preferences.

Consider the economy $\mathcal{E}$, an FME $\big((\bar{x}^h, \bar{\theta}^h)_{h=1,\dots,H}, \bar{q}\big)$, and a marginal non-redundant income stream $y \in \mathbb{R}^S \backslash \langle A \rangle$. If utility functions are differentiable, it follows (Magill and Quinzii 1996, Proposition 12.4) that

$$\bar{q} = \bar{\pi}_{\langle A \rangle} A,$$

where

$$\bar{\pi}_{\langle A \rangle} \equiv \pi^1(\bar{x}^1)_{\langle A \rangle} = \cdots = \pi^H(\bar{x}^H)_{\langle A \rangle},$$

and

$$\pi^h(\bar{x}^h)_s = \frac{\partial u^h(\bar{x}^h)/\partial x_s}{\partial u^h(\bar{x}^h)/\partial x_0}, \quad s = 1, \dots, S; h = 1, \dots, H,$$

[11] A crucial assumption for this result might be that options are in zero supply. If options are in unit supply (managerial stock options for example) then they change initial endowments and might give their owners a possibility to insure non-owners against risk without a loss of utility to themselves.

is the present value of state s to agent h. By analogy, the equilibrium price of the new asset y in the economy $\tilde{\mathcal{E}}$ equals

$$\tilde{q}_{J+1} = \tilde{\pi}_{\langle \tilde{A} \rangle} y = (\pi_{\langle A \rangle} + d\pi) y = \pi_{\langle A \rangle} y_{\langle A \rangle} + (d\pi)^\top y = RPV(y) + (d\pi)^\top y.$$

So, the difference between the replicating portfolio value and the equilibrium value of the new asset is determined by the change in the orthogonal projection of the agents' present value vectors on $\langle \tilde{A} \rangle$. For each agent h the introduction of the marginal asset y leads to a change in the present value of state s. For CRRA and CARA preferences this change is given by

$$d\pi_s^h = \begin{cases} -\gamma^h \delta p_s^h \left(\frac{\bar{x}_s^h}{\bar{x}_0^h}\right)^{-\gamma^h - 1} d\left(\frac{x_s}{x_0}\right) \\ -\gamma^h \delta p_s^h e^{-\gamma^h (\bar{x}_s^h - \bar{x}_0^h)} d(x_s - x_0) \end{cases} = \begin{cases} -\gamma^h \delta p_s^h \pi^h (\bar{x}^h)_s \frac{\bar{x}_0^h}{\bar{x}_s} d\left(\frac{x_s}{x_0}\right) \\ -\gamma^h \delta p_s^h \pi^h (\bar{x}^h)_s d(x_s - x_0), \end{cases}$$

respectively.

In other words, the change in the present value vector for agent h depends on the relative (CRRA) or the absolute (CARA) change in consumption. Assuming that current portfolios (in $\mathcal{E}$) do not change due to the introduction of the marginal asset y, it holds that

$$x_s^h = \omega_s^h + A_s \bar{\theta}^h + y_s \theta_{J+1}^h = \bar{x}_s^h + y_s \theta_{J+1}^h, \quad \text{and}$$
$$x_0^h = \omega_0^h - q\bar{\theta}^h - q_{J+1} \theta_{J+1}^h = \bar{x}_0^h - q_{J+1} \theta_{J+1}^h.$$

This implies that, for CRRA preferences, the change in the present value of state s is linear in

$$\frac{\bar{x}_0^h}{\bar{x}_s^h} d\left(\frac{x_s}{x_0}\right) = \frac{\bar{x}_0^h}{\bar{x}_s^h} \frac{\bar{x}_0^h y_s + \bar{x}_s^h q_{J+1}}{(\bar{x}_0^h)^2} d\theta_{J+1}^h = \left(\frac{y_s}{\bar{x}_s^h} + \frac{q_{J+1}}{\bar{x}_0^h}\right) d\theta_{J+1}^h,$$

whereas for CARA preferences the change is linear in

$$d(x_s^h - x_0^h) = (y_s + q_{J+1}) d\theta_{J+1}^h.$$

A check[12] reveals that in the simulations with CRRA preferences it typically holds that $d_s^3 < \bar{x}_s^h$ and $GEV(d^3) < \bar{x}_0^h$. Therefore, as a first order approximation, the change in present value vectors relative to the equilibrium one in $\mathcal{E}$ due to the introduction of the option is smaller for CRRA preferences than for CARA preferences. So, $(d\pi)^\top d^3$ is (by approximation) smaller for CRRA than for CARA preferences, implying that $RPV(d^3)$ is a better approximation for $GEV(d^3)$ in the CRRA than in the CARA case. It is important to reiterate that this effect is due to the fact that an agent with CRRA preferences is interested in changes in her relative position, whereas

12 This has been checked for the model with log-normal returns and homogeneous beliefs.

a CARA agent cares about absolute changes. Interestingly, this is exactly the same reason that often leads to the use of a CARA specification. An example in case is the literature on incomplete market dynamics in infinite horizon calibrated (production) economies (cf. Angeletos and Calvet 2006 and references therein). From the point of view of asset pricing, the dependence on wealth in the CRRA specification is desirable as it implies smaller deviations from the two fundamental theorems of finance in complete markets.

6 Conclusions

Over the past decades, the literature on asset pricing has largely developed separately from the literature on optimal portfolio choice. For a large part this is due to the two fundamental theorems of asset pricing. The first fundamental theorem tells us that, in the absence of arbitrage opportunities, asset prices are a linear combination of the assets' dividends, its coefficients being called the state prices. The second theorem, in addition, establishes that if markets are complete then these state prices are unique. This has led to an entire industry which computes the value of new assets based on the assumptions of no-arbitrage and market completeness. The big advantage of these assumptions is that asset prices can be computed without any knowledge of the agents' preferences and endowments. Since each new asset is redundant in complete markets, there exists a unique replicating portfolio, the value of which should be the no-arbitrage price of the new asset.

If, however, markets are incomplete, then the vector of state prices is not uniquely determined and there is no replicating portfolio. As a result, there is no clear-cut way in which to obtain *the* arbitrage-free price of a new—and as yet non-traded—asset. Instead, there is a plethora of prices, each supported by a state price vector which does not admit arbitrage opportunities. If one still assumes that in such circumstances the orthogonal projection of the dividends of the new asset on the old market span is the replicating portfolio, then one risks making a systematic error in determining the asset's value.

A simulation study under several distributional assumptions reveals two interesting effects of the introduction of a call option in a two-period model with incomplete markets. Firstly, for all specifications the change in the equilibrium price of the underlying asset is statistically insignificant (at the 1% level). This is, however, at odds with several empirical papers, which show that the introduction of large batches of options has, in fact, significantly changed the prices of the underlying. This indicates that (two-period) models with exogenous option payoffs should be used with care when studying option pricing. In reality, namely, option payoffs are endogenously determined by the underlying asset's equilibrium price. This could provide a channel through which asset prices might change after the introduction of options.

Secondly, it appears that the replicating portfolio value of options is a better approximation for its true, equilibrium, value if agents have CRRA rather than CARA preferences. The replicating portfolio value is attractive from a practical viewpoint since it basically assumes that the two fundamental theorems of finance are also valid in incomplete markets. The difference might be due to the fact that a CRRA agent cares about relative changes in consumption, whereas a CARA agent cares about absolute changes. It is exactly this argument that leads many researchers to use CARA preferences when calibrating infinite horizon incomplete market models.

Appendix

A. Homotopy methods in GEI analysis

The equilibria in this paper are computed by using a homotopy method developed in Herings and Kubler (2002). This is a differentiable homotopy obtained by replacing excess demand functions by the first order conditions of utility maximisation, an approach proposed by Garcia and Zangwill (1981). The advantage of this approach is that the number of agents, H, and the number of assets, J, determine the dimensionality of the homotopy instead of the number of states, S, which is typically very large. Furthermore, there is no need to explicitly compute agents' demand function and the set of no-arbitrage prices Q, both typically non-trivial. Instead one merely needs the Jacobians of the utility functions.

The Herings–Kubler (HK) homotopy is designed for two-period GEI economies with no consumption at $t = 0$. A standard way of transforming any GEI economy $\mathcal{E}$ to an economy with no consumption at time 0 is presented in Hens (1991) and consists of replacing the asset payoff matrix A by the matrix

$$\bar{A} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & A_1^1 & \cdots & A_1^J \\ \vdots & \vdots & \ddots & \vdots \\ 0 & A_S^1 & \cdots & A_S^J \end{bmatrix}.$$

That is, $t = 0$ consumption is translated to $t = 1$. by introducing an artificial state $s = 0$ and an artificial asset $j = 0$. The analysis that follows is concerned with the economy $\bar{\mathcal{E}} = (u, \omega, \bar{A})$.

The following additional assumption is made with respect to investors' preferences.

Assumption 1 For all $h = 1, \ldots, H$, the utility function u^h is three times continuously differentiable such that for all $x \in \mathbb{R}^S_{++}$ it holds that

1. $\partial u^h(x) \in \mathbb{R}^S_{++}$;
2. $\forall_{y \neq 0 : \partial u^h(x) y = 0} : y^\top \partial^2 u^h(x) y < 0$;
3. $\{\tilde{x} \in \mathbb{R}^S_{++} | u^h(\tilde{x}) \geq u^h(x)\}$ is closed in $\mathbb{R}^S$.

Let $\hat{f}$ and $\hat{q}$ denote the excess demand function and an asset price system, respectively, with the first entry removed. The algorithm starts from an initial price system

q^0 in Q, with the price of $t = 0$ consumption normalised to 1. Equilibria of $\bar{\mathcal{E}}$ are computed by the homotopy $\mathcal{H} : [0, 1] \times Q \to \mathbb{R}^J$

$$\mathcal{H}(t, q) = t\hat{f}(q) + (1 - t)(\hat{q}^0 - \hat{q}). \tag{12}$$

Herings and Kubler (2002) prove the following theorem.

Theorem 1 *Let $\Omega \subset \mathbb{R}^{HS}_{++}$ be an open set with full Lebesgue measure. For all initial endowments $\omega \in \Omega$ it holds that*

1. *$\mathcal{H}^{-1}(\{0\})$ is a compact C^2 one-dimensional manifold with boundary $\mathcal{H}^{-1}(\{0\}) \cap (\{0, 1\} \times Q)$;*
2. *there is an odd number of solutions in $\mathcal{H}^{-1}(\{0\}) \cap (\{1\} \times Q)$; itemthere is one solution in $\mathcal{H}^{-1}(\{0\}) \cap (\{0\} \times Q)$;*
3. *there is no sequence $(t^n, q^n)_{n\in\mathbb{N}}$ in $\mathcal{H}^{-1}(\{0\})$ with limit $(t, q) \in [0, 1] \times \partial Q$ or such that $||(t^n, q^n)||_2 \to \infty$.*

That is, generically, there exists a path from q^0 to an FME q; there is only one solution at q^0; there is an odd number of solutions; and the algorithm does not diverge or converge to the boundary.

The homotopy $\mathcal{H}$ has the advantage that one does not have to compute the set Q explicitly. Unfortunately, however, it is usually non-trivial to compute the excess demand function f analytically. One can use (12), but at every step n, the function value $\hat{f}(q^n)$ has to be computed numerically, which is highly time consuming. Instead one can replace $\mathcal{H}$ with the diffeomorphic implementable homotopy $\mathcal{H}^* : [0, 1] \times Q \times \mathbb{R}^{H(J+1)\times\mathbb{R}^H} \to \mathbb{R}^{(H+1)(J+2)-2}$, defined by

$$\mathcal{H}^*(t, q, \theta, \lambda) = \begin{cases} t\sum_{h=1}^{H}\theta_j^h + (1-t)(q_j^0 - q_j), & j = 1, \ldots, J \\ \left(\partial u^h(\omega^h + \bar{A}\theta^h)\bar{A}\right)^{\top} - \lambda^h q^{\top}, & h = 1, \ldots, H \\ q\theta^h, & h = 1, \ldots, H, \end{cases} \tag{13}$$

where λ is the vector of Lagrange multipliers obtained from utility maximisation. We have the following result.

Theorem 2 (Herings and Kubler 2002) *$(\mathcal{H}^*)^{-1}(\{0\})$ is C^2-diffeomorphic to $\mathcal{H}^{-1}(\{0\})$.*

This implies that, generically, the homotopy $\mathcal{H}^*$ converges to an FME. The homotopy (13) is implemented in Matlab via a four-step Adams–Bashforth predictor-corrector method (cf. Lambert 1991).

B. Empirical results of simulations

In all Tables 1–5, z_α denotes the z-value of (5), z_{EV^h} denotes the z-value of (6), all $h = 1, \ldots, H$, z_{q_j} denotes the z-value of (7), all $j = 1, \ldots, J$, and z_{RPV} denotes the z-value of (8). Significance at the 10%, 5%, and 1% level is indicated by $*$, $**$, and $***$, respectively.

Table 1 Results from the simulations with tri-nomial shocks

	z_α	z_{EV^1}	z_{EV^2}	z_{EV^3}	z_{q_1}	z_{q_2}	$z_{q_{RPV}}$
CARA	0.2775	0.0597	−0.2999	0.2537	−1.3287	−0.2718	3.7077***
CRRA	−1.4840	1.0672	1.0050	3.1744***	−1.8498*	−0.5368	1.7521*

Table 2 Results from the simulations with uniform shocks

	z_α	z_{EV^1}	z_{EV^2}	z_{EV^3}	z_{q_1}	z_{q_2}	$z_{q_{RPV}}$
CARA	6.5163***	3.5240***	4.0834***	1.0051	−1.5128	−0.4182	2.9002***
CRRA	5.7539***	1.0924	1.1292	−0.9379	−2.2863**	−0.0975	0.8231

Table 3 Results from the simulations with log-normal shocks

	z_α	z_{EV^1}	z_{EV^2}	z_{EV^3}	z_{q_1}	z_{q_2}	$z_{q_{RPV}}$
CARA	5.3582***	0.0925	6.2474***	1.1347	0.2992	1.0109	3.7591***
CRRA	5.7658***	1.5532	2.8028***	2.2927**	−1.3780	−0.9918	0.1496

Table 4 Results from the simulations with log-normal shocks and downward jumps

	z_α	z_{EV^1}	z_{EV^2}	z_{EV^3}	z_{q_1}	z_{q_2}	$z_{q_{RPV}}$
CARA	9.5549***	3.0290***	1.0052	2.6206***	−1.9190*	0.1150	3.8904***
CRRA	10.5868***	1.6352	1.0077	1.1950	−1.0112	−1.6057	0.6204

Table 5 Results from the simulations with log-normal shocks, downward jumps, and heterogeneous beliefs

	z_α	z_{EV^1}	z_{EV^2}	z_{EV^3}	z_{q_1}	z_{q_2}	$z_{q_{RPV}}$
CARA	11.6678***	1.6561*	1.0051	1.0050)	0.6767	0.1511	2.8618***
CRRA	11.5432***	2.3374**	2.7879***	2.6024***	−1.2342	1.2020	0.1424

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